Estimate of the Three-Loop $\overline{MS}$ Contribution to $\sigma(W_L^+ W_L^- \to Z_L Z_L)$

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Abstract

The three-loop contribution to the $\overline{MS}$ single-Higgs-doublet standard-model cross-section $\sigma(W_L^+ W_L^- \to Z_L Z_L)$ at $s = (5M_H)^2$ is estimated via least-squares matching of the asymptotic Padé-approximant prediction of the next order term, a procedure that has been previously applied to QCD corrections to correlation functions and decay amplitudes. In contrast to these prior applications, the expansion parameter for the $W_L^+ W_L^- \to Z_L Z_L$ process is the non-asymptotically-free quartic scalar-field coupling of the standard model, suggesting that the least-squares matching be performed over the "infrared" $\mu^2 \leq s$ region of the scale parameter. All three coefficients of logarithms within the three-loop term obtained by such matching are found to be within 6.6% relative error of their true values, as determined via renormalization-group methods. Surprisingly, almost identical results are obtained by performing the least squares matching over the $\mu^2 \geq s$ region.

Within standard-model single-Higgs-doublet electroweak physics, the cross-section for the scattering of two longitudinal W's into two longitudinal Z's at very high energy takes the form

$$\sigma[s, L(\mu), g(\mu)] = \frac{8\pi^3}{9s} H[s, L(\mu), g(\mu)], \tag{1}$$

where the scale-sensitive portion of the cross-section [1],

$$\begin{aligned} H[s, L(\mu), g(\mu)] &= g^2(\mu)\{1 + [-4L(\mu) - 10.0]g(\mu) \\ &+ \left[12L^2(\mu) + 68.667L(\mu) + \left(93.553 + \frac{2}{3}ln\left(\frac{s}{M_H^2}\right)\right)\right] g^2(\mu) \\ &+ \left[c_3 L^3(\mu) + c_2 L^2(\mu) + c_1 L(\mu) + c_0\right] g^3(\mu) + ...\}, \end{aligned} \tag{2}$$

depends on the renormalization scale μ explicitly through the logarithm

$$L(\mu) \equiv ln(\mu^2/s) \tag{3}$$

and implicitly through the $\overline{MS}$ quartic scalar-field coupling

$$g(\mu) = 6\, \lambda_{\overline{MS}}(\mu)/16\pi^2. \tag{4}$$

The three-loop coefficients $\{c_0, c_1, c_2, c_3\}$ in (2) are presently unknown. The factor of M_H appearing explicitly in the two-loop term calculated in [1] is a scale-independent pole mass; the coefficients $\{c_0, c_1, c_2, c_3\}$ can also exhibit dependence on this mass without acquiring additional μ-dependence.

In this note, we utilise asymptotic Padé-approximant methods to predict these four coefficients. Such methods have already been applied to predicting next-order terms within the renormalization group functions of QCD [2,3,4], supersymmetric QCD [2,5], and massive scalar field theory [3,4,6], as well as next-order QCD corrections to scalar and vector current correlation functions [4,7] and various decay processes [8,9,10]. In all of these applications, the Padé-estimation procedures are tested (with surprising success) against either those higher-order terms already known from explicit calculation, such as renormalization group functions in scalar field theories [11,12], QCD [13], and supersymmetric QCD [14], or against those coefficients of logarithms [such as $\{c_1, c_2, c_3\}$ in (2)] which can be extracted via renormalization group methods [7,8,9,10]. Below, we shall apply the latter testing procedure to estimates of the next-order terms $\{c_1, c_2, c_3\}$ in the $\overline{MS}$ cross-section (1).

Given a perturbative series of the form $1 + R_1 g + R_2 g^2 + R_3 g^3 + ...$ where R_3 is not known, as is the case in (2), asymptotic Padé-approximant methods can be employed to show that [4]

$$R_3 \cong \frac{2R_2^3}{R_1^3 + R_1 R_2}. \tag{5}$$

This result is, of course, contingent upon the field-theoretical series exhibiting appropriate asymptotic behaviour. Its derivation (explicitly presented in [8]) follows from the $O(1/N)$ error anticipated from an $[N|1]$ Padé-approximant prediction of R_{N+2} [15], a semi-empirical behaviour which is seen to characterise a number of field-theoretical applications even when N is small.

For the case of the series (2), however, R_1 is linear in L and R_2 is quadratic in L. Consequently, the prediction (5) for R_3 corresponds to a rational function of L incompatible with the degree-3 polynomial in L anticipated from (2). Clearly, a procedure is required by which predictions for the polynomial coefficients $\{c_0, c_1, c_2, c_3\}$ in (2) can be extracted from (5). In past applications where the same problem arises [7,8,9,10], one method employed is a least-squares matching of (5) to the form $R_3 = c_0 + c_1 L(\mu) + c_2 L^2(\mu) + c_3 L^3(\mu)$ over the full perturbative domain of μ. For QCD calculations this domain is ultraviolet; *e.g.* in estimating three-loop QCD corrections to $B \to X_u \ell^- \overline{\nu}_\ell$ the matching is over the ultraviolet domain $\mu \geq m_b(\mu)$ [8]. For the expression (2), in which the perturbative expansion parameter is the non-asymptotically-free quartic scalar coupling $\lambda_{\overline{MS}}(\mu)$, the appropriate domain for such a least-squares matching is *infrared.*

Thus, to obtain predicted values for $\{c_0, c_1, c_2, c_3\}$ for a given choice of M_H, we choose a least-squares matching over the region $\mu^2 \leq s$, or alternatively $0 < w \leq 1$, where $w \equiv \mu^2/s$ is the argument of the logarithm (3). From (2) and (5), this matching is achieved by minimizing the function

$$\chi^2(c_0, c_1, c_2, c_3) \equiv \int_{w_{min}}^{1} dw \left[\frac{2R_2^3(w)}{R_1^3(w) + R_1(w)R_2(w)} - c_0 - c_1 ln(w) - c_2 ln^2(w) - c_3 ln^3(w) \right]^2 \quad (6)$$

with respect to c_0, c_1, c_2, c_3, where $R_1(w)$ and $R_2(w)$ are explicitly given in (2):

$$R_1(w) = -4ln(w) - 10.0, \quad (7)$$

$$R_2(w) = 12ln^2(w) + 68.667ln(w) + \left(93.553 + \frac{2}{3} ln\left(\frac{s}{M_H^2}\right)\right). \quad (8)$$

The lower bound of integration w_{min} in (6) would ordinarily be zero to encompass the full $\mu^2 \leq s$ range. However, we are compelled to consider a nonzero value of w_{min} in order to avoid any integrand poles, as discussed below. The expressions (1) and (2) are stated in ref. [1] to be accurate (within single-digit percent errors) only in the high-energy limit $\sqrt{s} \gtrsim 5M_H$. Although the projected linear and quadratic dependence of c_1 and c_0 on $ln(s/M_H^2)$ could, in principle, be extracted via Padé methods,[1] the relatively small coefficient of this logarithm in the previous-order term (8) necessarily implies a similar insensitivity to this logarithm in Padé estimates of next-order terms. Consequently, we restrict our analysis here to the $s = (5M_H)^2$ kinematic boundary of applicability for (1) and (2). With this choice, the integrand of (6) acquires singularities at 0.0552, 0.0821, and 0.0896. Consequently, we choose $w_{min} = 0.09$ to include virtually all of the integrable infrared region, and we find that

$$\begin{aligned} \chi^2(c_0, c_1, c_2, c_3) & \\ = & \ 248033 + 813.779c_0 - 332.772c_1 + 244.574c_2 - 242.664c_3 \\ + & \ 0.91c_0^2 - 1.38657c_0c_1 + 1.72946c_0c_2 \\ - & \ 2.67527c_0c_3 + 0.864732c_1^2 - 2.67527c_1c_2 + 4.64965c_1c_3 \\ + & \ 2.32482c_2^2 - 8.67669c_2c_3 + 8.48631c_3^2 \end{aligned} \quad (9)$$

By then optimizing (9) with respect to c_0, c_1, c_2, c_3, we obtain the following Padé predictions for these coefficients:

$$c_0^{Padé} = -896, c_1^{Padé} = -889, c_2^{Padé} = -288, c_3^{Padé} = -30.5. \quad (10)$$

As noted earlier, the true values of the coefficients c_1, c_2, c_3 can be extracted via the scale-invariance of the physical cross-section (1):

$$O = \mu^2 \frac{dH}{d\mu^2}[s, L(\mu), g(\mu)] = \frac{\partial H}{\partial L} + \beta(g) \frac{\partial H}{\partial g}, \quad (11)$$

where [11]

$$\beta(g) = 2g^2 - \frac{13}{3} g^3 + 27.803g^4 + \dots . \quad (12)$$

[1] In ref. [10], the polynomial dependence of three-loop order terms in $H \to gg$ on the logarithm of the pole-mass ratio M_H/M_t is similarly extracted.

One can verify that the known terms in (2) satisfy the renormalization-group equation (11) to $O(g^3)$ and $O(g^4)$, as is evident from the series expansions

$$\frac{\partial H}{\partial L} = -4g^3 + (24L + 68.667)g^4 + (3c_3L^2 + 2c_2L + c_1)g^5 + ... , \tag{13}$$

$$\begin{aligned} \beta(g)\frac{\partial H}{\partial g} &= 4g^3 + (-24L - 68.667)g^4 + (96L^2 + 601.33L + 934.028 \\ &+ 5.333ln(s/M_H^2))g^5 + ... \end{aligned} \tag{14}$$

We find upon incorporating $O(g^5)$ terms of (13) and (14) into the right-hand side of (11) that

$$\begin{aligned} c_1 &= -934.028 - 5.333ln(s/M_H^2) \underset{s=25M_H^2}{\longrightarrow} -951.2, \\ c_2 &= -300.67,\ c_3 = -32. \end{aligned} \tag{15}$$

The Padé predictions (10) for c_1, c_2, c_3 are respectively seen to be within relative errors of 6.6%, 4.3%, and 4.7% of their true values (15).

Curiously, the accuracy of these Padé results does not appear to be contingent upon the matching being performed over the "infrared" $\mu^2 < s = 25\ M_H^2$ range, as motivated by the non-asymptotically free character of the scalar field coupling $g(\mu)$. Indeed, a potential drawback of fitting over the $\mu^2 \leq s$ (hence, $w \leq 1$) region, as in (6), is the negativity of $ln(w)$ over the entire range of integration. Since the c_i ultimately obtained in (10) are all same-sign (negative), cancellations necessarily occur between successive $c_k ln^k(w)$ terms in the integrand of (6) in the best-fit region of c_k parameter-space. To address this issue, we have also performed a fit of the Padé-prediction (5) to the third-subleading order of (2) over the entire $\mu^2 \geq s$ (*i.e.* $w \geq 1$) region in which $ln(w)$ is *positive.* This entails integration of the integrand of (6) with appropriately modified bounds of integration to encompass the ultraviolet region:

$$\int_{w_{min}}^{1} dw[...]^2 \rightarrow \int_{1}^{\infty} dw[...]^2. \tag{16}$$

We then find that

$$\begin{aligned} \chi^2(c_0, c_1, c_2, c_3) &= 1.45665 \cdot 10^7 + 5095.28c_0 + 10294.6c_1 \\ &+ 35519.6c_2 + 167166c_3 + c_0^2 + 2c_0c_1 \\ &+ 4c_0c_2 + 12c_0c_3 + 2c_1^2 + 12c_1c_2 + 48c_1c_3 \\ &+ 24c_2^2 + 240c_2c_3 + 720c_3^2, \end{aligned} \tag{17}$$

which upon optimization, yields values for $c_k[c_0 = -896,\ c_1 = -889, c_2 = -289\ c_3 = -30.9]$ that are virtually the same as those listed in (10). The

small relative errors characterising our Padé estimates of c_1, c_2, c_3 are comparable to those characterising Padé estimates of renormalization-group accessible coefficients within next-order QCD corrections to other processes [7,8,9,10], and suggest similar accuracy in the estimated value of the renormalization-group *inaccessible* three-loop coefficient c_0 in (10).

We therefore conclude that Padé-approximant predictions of the next order contribution to $WW \rightarrow ZZ$ at very high energies appear to be consistent and reliable. It should also be noted that higher order β-function terms associated with the evolution of the quartic scalar-field coupling constant (4) are themselves accurately predicted by the same asymptotic Padé-approximant methods employed above for $\sigma(WW \rightarrow ZZ)$. If we express the β-function (12) in the form

$$\beta(g) = 2g^2(1 + R_1 g + R_2 g^2 + R_3 g^3 + ...); \quad R_1 = -13/6, \; R_2 = 13.915, \quad (18)$$

we predict via (5) that $R_3 = -133.6$, or alternatively, that the predicted next term in the series (12) is $-267.2g^5$. This is quite close to $-266.495g^5$, the true calculated value [11] of the next-order β-function contribution. Similarly close agreement between the somewhat more complicated asymptotic Padé-approximant prediction and the explicit calculation of the $O(g^6)$ contribution to this β-function is demonstrated in ref. [6].

Thus, the results presented above are an example of how Padé estimation procedures can anticipate next-order contributions whose exact values are obtainable only by lengthy calculation. For the particular process in question, the distinction between two- and three-loop order results is seen to be unimportant unless the mass of the (Salam-Weinberg) Higgs field mediating the scattering process is very large. This is illustrated in Figures 1-3, which compare two loop and three loop expressions for the scale sensitive portion (2) of the $WW \rightarrow ZZ$ cross section (1) for Higgs-field masses of 200, 400, and 600 GeV, respectively. Only for the largest of these three choices is an appreciable difference anticipated between two- and three-loop order predictions for the cross section.

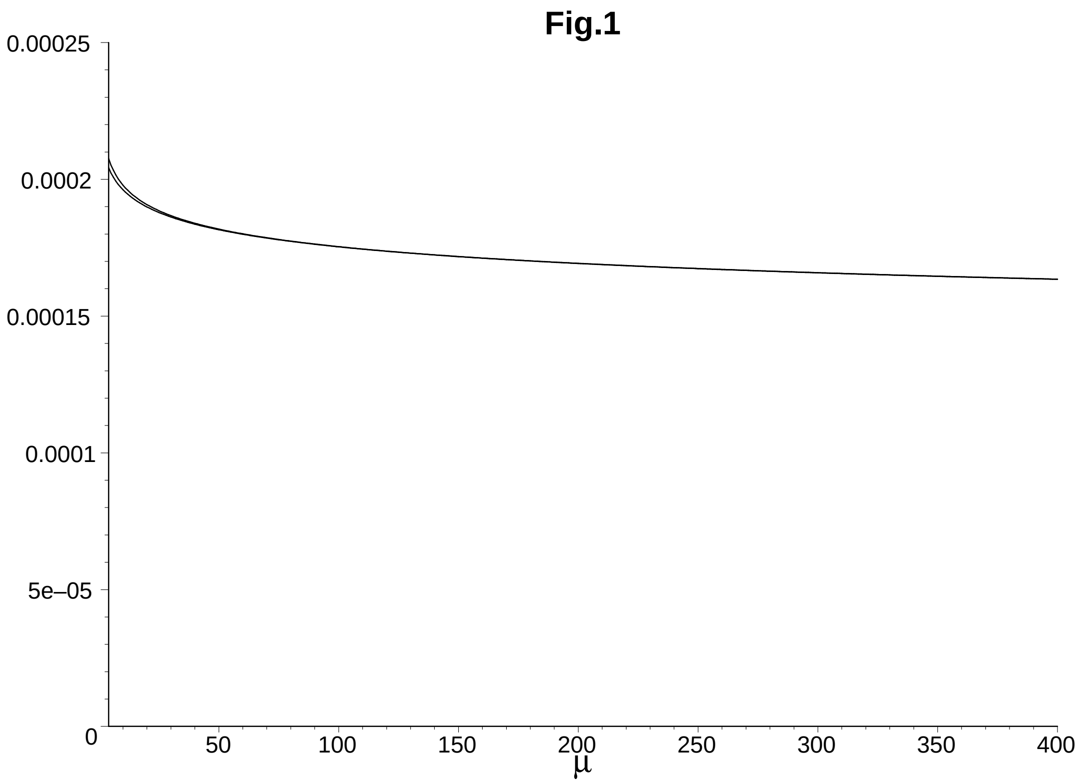

Figure 1: The two-loop (bottom curve) and predicted three-loop (top curve) expressions for the scale-sensitive portion $H[s = (5M_H)^2, L(\mu), g(\mu)]$ of the cross-section (1) are plotted for Higgs mass $M_H = 200$ GeV.

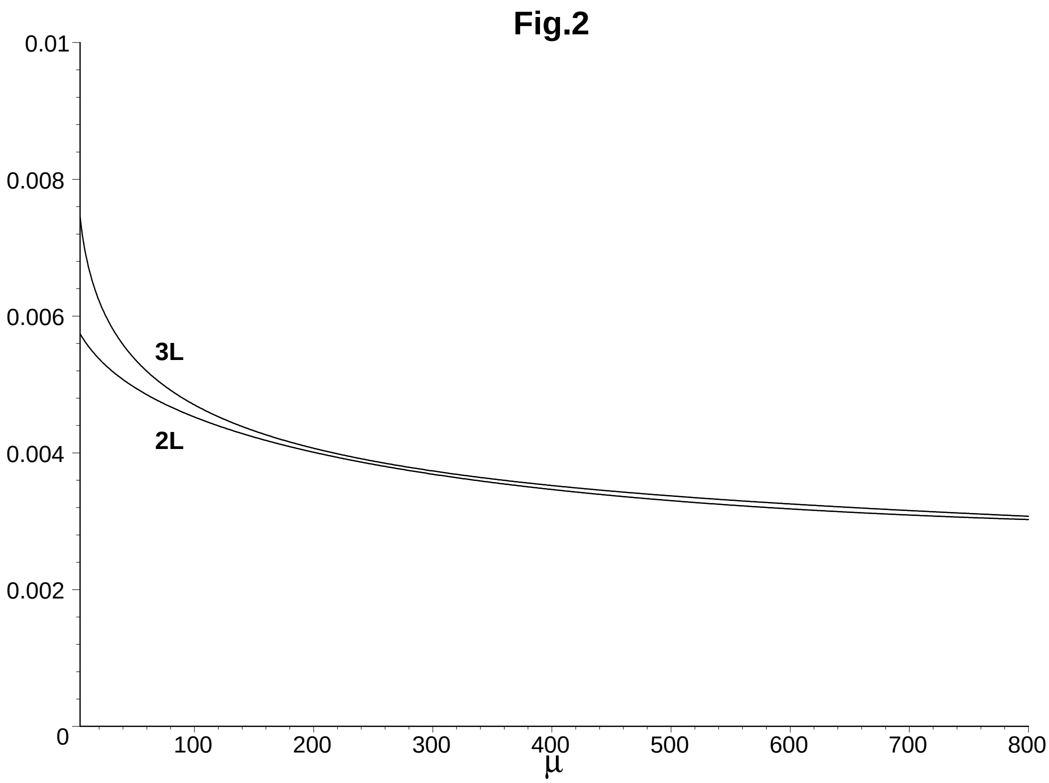

Figure 2: Comparison of two-loop (2L) and three-loop (3L) expressions, as in Figure 1, but with $M_H = 400$ GeV.

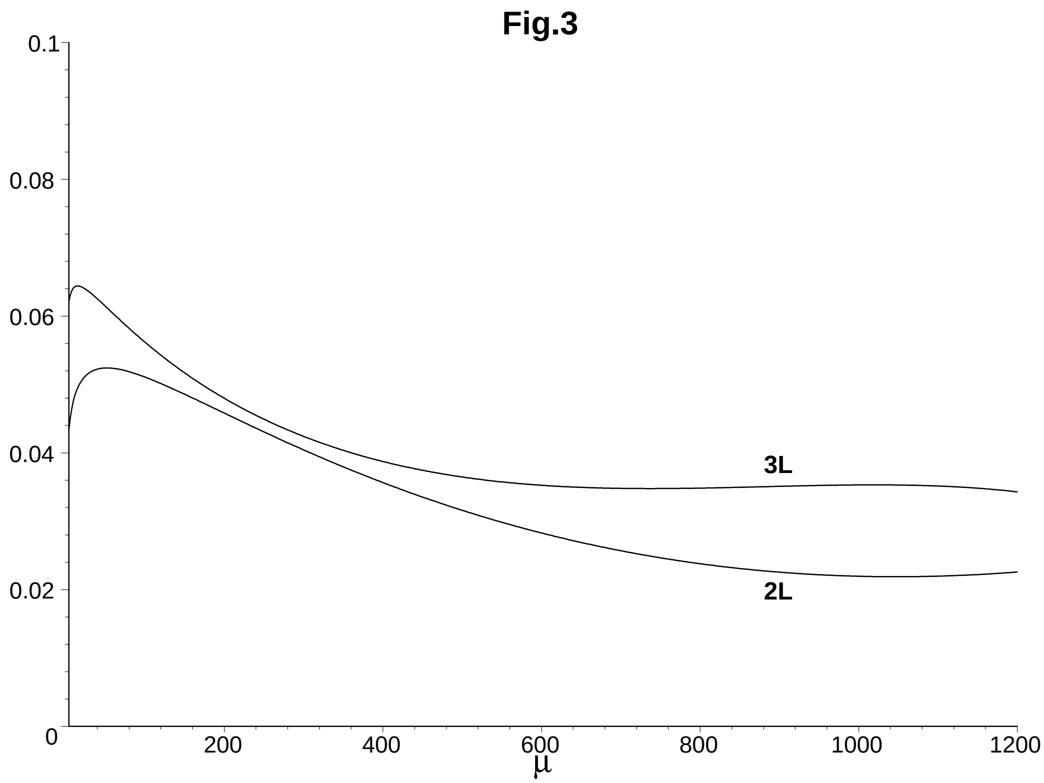

Figure 3: Comparison of two-loop (2L) and three-loop (3L) expressions, as in Figure 1, but with $M_H = 600$ GeV.

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